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# Conditions for Choquet Integral Representation

(Choquet 積分表現のための条件)

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## 1 Introduction

The Choquet integral with respect to a fuzzy measure is a functional on the class  $B$  of measurable functions, that is comonotonically additive and monotone (for short c.m.).

Sugeno et al. [15] proved that a c.m. functional  $I$  can be represented by a Choquet integral with respect to a regular fuzzy measure when the domain of  $I$  is the class  $K^+$  of nonnegative continuous functions with compact support on a locally compact Hausdorff space. In [8, 9], it is proved that a c.m. functional is a rank- and sign-dependent functional, that is, the difference of two Choquet integrals. This functional is used in utility theory [5] and cumulative prospect theory [17, 18]. It is also proved that a rank- and sign-dependent functional is a c.m. functional if the universal set  $X$  is not compact.

In this paper, we discuss the conditions for which a c.m. functional can be represented by one Choquet integral. We define the conjugate conditions and show their basic proper-

ties in Section 4. The conjugate conditions are stronger than the boundedness and a c.m. functional  $I$  is represented by one Choquet integral when  $I$  satisfies one of the conjugate conditions. Conversely if a c.m. functional  $I$  is represented by one Choquet integral,  $I$  satisfies the conjugate condition when the universal set  $X$  is separable.

The proof of the main theorem is shown in Section 5 and the other proofs are omitted.

## 2 Preliminaries

In this section, we define the fuzzy measure, the Choquet integral and the rank- and sign-dependent functional, and show their basic properties.

Throughout the paper we assume that  $X$  is a locally compact Hausdorff space,  $\mathcal{B}$  is the class of Borel subsets,  $\mathcal{O}$  is the class of open subsets and  $\mathcal{C}$  is the class of compact subsets.

**Definition 2.1.** [14] A *fuzzy measure*  $\mu$  is an extended real valued set function,

$$\mu : \mathcal{B} \longrightarrow \overline{R^+}$$

with the following properties,

$$(1) \mu(\emptyset) = 0$$

$$(2) \mu(A) \leq \mu(B) \text{ whenever } A \subset B, A, B \in \mathcal{B}$$

where  $\overline{R^+} = [0, \infty]$  is the set of extended nonnegative real numbers.

When  $\mu(X) < \infty$ , we define the *conjugate*  $\mu^c$  of  $\mu$  by

$$\mu^c(A) = \mu(X) - \mu(A^c)$$

for  $A \in \mathcal{B}$ .

**Definition 2.2.** Let  $\mu$  be a fuzzy measure on measurable space  $(X, \mathcal{B})$ .

$\mu$  is said to be *outer regular* if

$$\mu(B) = \inf\{\mu(O) | O \in \mathcal{O}, O \supset B\}$$

for all  $B \in \mathcal{B}$ .

The outer regular fuzzy measure  $\mu$  is said to be *regular*, if for all  $O \in \mathcal{O}$

$$\mu(O) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset O\}.$$

We denote by  $K$  the class of continuous functions with compact support, by  $K^+$  the class of nonnegative continuous functions with compact support and by  $K_1^+$  the class of nonnegative continuous functions with compact support that satisfies  $0 \leq f \leq 1$ .

We denote  $\text{supp}(f)$  the support of  $f \in K$ , that is,

$$\text{supp}(f) = \text{cl}\{x | f(x) \neq 0\}.$$

**Definition 2.3.** [1, 6] Let  $\mu$  be a fuzzy measure on  $(X, \mathcal{B})$ .

(1) The *Choquet integral* of  $f \in K^+$  with respect to  $\mu$  is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where  $\mu_f(r) = \mu(\{x | f(x) \geq r\})$ .

(2) Suppose  $\mu(X) < \infty$ . The Choquet integral of  $f \in K$  with respect to  $\mu$  is defined by

$$(C) \int f d\mu = (C) \int f^+ d\mu - (C) \int f^- d\mu^c,$$

where  $f^+ = f \vee 0$  and  $f^- = -(f \wedge 0)$ . When the right hand side is  $\infty - \infty$ , the Choquet integral is not defined.

**Definition 2.4.** [3] Let  $f, g \in K$ . We say that  $f$  and  $g$  are *comonotonic* if

$$f(x) < f(x') \Rightarrow g(x) \leq g(x')$$

for  $x, x' \in X$ . We denote  $f \sim g$ , when  $f$  and  $g$  are comonotonic.

**Definition 2.5.** Let  $I$  be a real valued functional on  $K$ .

We say  $I$  is *comonotonically additive* iff

$$f \sim g \Rightarrow I(f + g) = I(f) + I(g)$$

for  $f, g \in K^+$ , and  $I$  is *monotone* iff

$$f \leq g \Rightarrow I(f) \leq I(g)$$

for  $f, g \in K^+$ .

If a functional  $I$  is comonotonically additive and monotone, we say that  $I$  is a *c.m. functional*.

Suppose that  $I$  is a c.m. functional, then we have  $I(af) = aI(f)$  for  $a \geq 0$  and  $f \in K^+$ , that is,  $I$  is positive homogeneous.

### 3 Representation and Boundedness

**Definition 3.1.** Let  $I$  be a real valued functional on  $K$ .  $I$  is said to be a *rank- and sign-dependent functional* (for short a *r.s.d. functional*) on  $K$ , if there exist two fuzzy measures  $\mu^+, \mu^-$  such that for every  $f \in K$

$$I(f) = (C) \int f^+ d\mu^+ - (C) \int f^- d\mu^-$$

where  $f^+ = f \vee 0$  and  $f^- = -(f \wedge 0)$ .

When  $\mu^+ = \mu^-$ , we say that the r.s.d. functional is the Šipoš functional [13]. If the r.s.d. functional is the Šipoš functional, we have  $I(-f) = -I(f)$ .

If  $\mu^+(X) < \infty$  and  $\mu^- = (\mu^+)^c$ , we say that the r.s.d. functional is the Choquet functional.

**Theorem 3.2.** [8, 9] *Let  $I$  be a c.m. functional on  $K$ .*

(1) *We put*

$$\mu_I^+(O) = \sup\{I(f) | f \in K_1^+, \text{supp}(f) \subset O\},$$

*and*

$$\mu_I^+(B) = \inf\{\mu_I^+(O) | O \in \mathcal{O}, O \supset B\}$$

*for  $O \in \mathcal{O}$  and  $B \in \mathcal{B}$ .*

*Then  $\mu_I^+$  is a regular fuzzy measure.*

(2) *We put*

$$\mu_I^-(O) = \sup\{-I(-f) | f \in K_1^+, \text{supp}(f) \subset O\},$$

*and*

$$\mu_I^-(B) = \inf\{\mu_I^-(O) | O \in \mathcal{O}, O \supset B\}$$

*for  $O \in \mathcal{O}$  and  $B \in \mathcal{B}$ .*

*Then  $\mu_I^-$  is a regular fuzzy measure.*

(3) *A c.m. functional is a r.s.d functional, that is,*

$$I(f) = (C) \int (f \vee 0) d\mu_I^+ - (C) \int -(f \wedge 0) d\mu_I^-$$

*for every  $f \in K$ .*

(4) If  $X$  is compact, then a c.m. functional can be represented by one Choquet integral.

(5) If  $X$  is locally compact but not compact, then a r.s.d functional is a c.m. functional.

**Definition 3.3.** Let  $I$  be a c.m. functional on  $K$ . We say that  $\mu_I^+$  defined in Theorem 3.2 is the regular fuzzy measure induced by the positive part of  $I$ , and  $\mu_I^-$  the regular fuzzy measure induced by the negative part of  $I$ .

**Definition 3.4.** Let  $I$  be a real valued functional on  $K$ .

(1)  $I$  is said to be *bounded above* if there exists  $M > 0$  such that

$$I(f) \leq M\|f\|$$

for all  $f \in K$ .

(2)  $I$  is said to be *bounded below* if there exists  $M > 0$  such that

$$-M\|f\| \leq I(f)$$

for all  $f \in K$ .

(3)  $I$  is said to be *bounded* if  $I$  is bounded above and below.

**Proposition 3.5.** [8, 11] Let  $I$  be a c.m. functional on  $K$  and  $\mu_I^+$  and  $\mu_I^-$  the regular fuzzy measure induced by  $I$ .

(1)  $I$  is bounded above iff  $\mu_I^+(X) < \infty$ .

(2)  $I$  is bounded below iff  $\mu_I^-(X) < \infty$ .

**Proposition 3.6.** [10] Let  $X$  be separable and  $I$  be a c.m. functional on  $K$  that is bounded, and  $\mu_I^+$  and  $\mu_I^-$  the regular fuzzy measure induced by  $I$ .

- (1) If  $(C) \int f d\mu_I^+ = (C) \int f d(\mu_I^-)^c$  for all  $f \in K^+$ , then  $\mu_I^+(C) = (\mu_I^-)^c(C)$  for all  $C \in \mathcal{C}$ .
- (2) If  $(C) \int f d\mu_I^- = (C) \int f d(\mu_I^+)^c$  for all  $f \in K$ , then  $\mu_I^-(C) = (\mu_I^+)^c(C)$  for all  $C \in \mathcal{C}$ .

Proposition 3.6 says that if a c.m. functional  $I$  is Choquet integral with respect to  $\mu_I^+$  then we have  $\mu_I^-(C) = (\mu_I^+)^c(C)$  for every  $C \in \mathcal{C}$ . Since  $(\mu_I^+)^c$  is not always regular, it is not always true that  $\mu_I^- = (\mu_I^+)^c$ . That is,  $I$  is not always Choquet functional. See the example in [8].

## 4 Conjugate condition for compact sets

**Definition 4.1.** Let  $I$  be a c.m. functional and  $C \in \mathcal{C}$ .

- (1) We say that  $I$  satisfies the positive *conjugate condition* for  $C$  if there exists a positive real number  $M$  such that for any  $\epsilon > 0$  there exist  $f_1, f_2 \in K_1$  satisfying the next condition.

$$1_C \leq g_1 \leq f_1 \text{ and } f_2 \leq g_2 \leq 1_{C^c} \text{ with } \text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c \text{ imply}$$

$$|I(-g_1) - I(g_2) + M| < \epsilon$$

$$\text{for } g_1, g_2 \in K_1.$$

- (2) We say that  $I$  satisfies the negative *conjugate condition* for  $C$  if there exists a positive real number  $M$  such that for any  $\epsilon > 0$  there exist  $f_1, f_2 \in K_1$  satisfying the next condition.



$1_C \leq g_1 \leq f_1$  and  $f_2 \leq g_2 \leq 1_{C^c}$  with  $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$  imply

$$|-I(g_1) + I(-g_2) + M| < \epsilon.$$

for  $g_1, g_2 \in K_1$ .

Suppose that a c.m. functional  $I$  satisfies the positive conjugate condition for  $\emptyset$ . Let  $g_1(x) = 0$  for all  $x \in X$ . Since  $\emptyset \subset \text{supp}(g_1)$  and  $I(g_1) = 0$ , there exists  $M > 0$  and for any  $\epsilon > 0$  there exists  $f_2 \in K_1^+$  such that  $\text{supp}(f_2) \subset \text{supp}(g_2) \subset X$  implies

$$|-I(g_2) + M| < \epsilon.$$

Therefore we have the next proposition.

**Proposition 4.2.** *Let  $I$  be a c.m. functional.*

- (1) *If  $I$  satisfies the positive conjugate condition for  $\emptyset$ , then  $I$  is bounded above.*
- (2) *If  $I$  satisfies the negative conjugate condition for  $\emptyset$ , then  $I$  is bounded below.*

The next lemma follows from the definition of the induced regular fuzzy measure.

**Lemma 4.3.** *Let  $A \in \mathcal{B}$  and  $f \in K^+$ . Suppose that  $A \subset \{x | f \geq 1\}$ , then we have*

$$\mu_I^+(A) \leq I(f) \text{ and } \mu_I^-(A) \leq -I(-f).$$

Applying this lemma, we have the next theorem. The detail of the proof is in Section 5.

**Theorem 4.4.** *Let  $C \in \mathcal{C}$ ,  $I$  be a c.m. functional and  $\mu_I^+$  and  $\mu_I^-$  the regular fuzzy measure induced by  $I$ .*

- (1)  *$I$  satisfies the positive conjugate condition for every  $C \in \mathcal{C}$  if and only if*

$$\mu_I^-(C) = (\mu_I^+)^c(C)$$

*for every  $C \in \mathcal{C}$ .*

(2) *I satisfies the negative conjugate condition for  $C$  if and only if*

$$\mu_I^+(C) = (\mu_I^-)^c(C)$$

*for every  $C \in \mathcal{C}$ .*

Suppose that a c.m. functional  $I$  satisfies the positive conjugate condition for all  $C \in \mathcal{C}$ . It follows from Theorem 4.4 that

$$\begin{aligned} \mu_I^-(X) &= \sup\{\mu_I^-(C) | C \subset X\} \\ &= \sup\{(\mu_I^+)^c(C) | C \subset X\} \\ &= \sup\{\mu_I^+(X) - \mu_I^+(C^c) | C \subset X\} \leq \mu_I^+(X). \end{aligned}$$

Therefore we have the next corollary.

**Corollary 4.5.** *If a c.m. functional  $I$  satisfies the positive or negative conjugate condition for all  $C \in \mathcal{C}$ , then  $I$  is bounded.*

It follow from Theorem 4.4 that

$$\mu_I^-(\{x | f(x) \geq r\}) = (\mu_I^+)^c(\{x | f(x) \geq r\})$$

for all  $f \in K$  and  $r > 0$ . Therefore we have the next theorem.

**Theorem 4.6.** *Let  $I$  be a c.m. functional.*

(1) *If  $I$  satisfies the positive conjugate condition for all  $C \in \mathcal{C}$ , we have*

$$I(f) = (C) \int f d\mu_I^+$$

*for all  $f \in K$ .*

(2) If  $I$  satisfies the negative conjugate condition for all  $C \in \mathcal{C}$ , we have

$$I(f) = -(C) \int -f d\mu_I^-$$

for all  $f \in K$ .

The next theorem follows from Proposition 3.6

**Theorem 4.7.** *Let  $X$  be separable and  $I$  be a c.m. functional on  $K$  that is bounded, and  $\mu_I^+$  and  $\mu_I^-$  the regular fuzzy measure induced by  $I$ .*

(1) *If  $I(f) = (C) \int f d\mu_I^+$  for all  $f \in K$ , then  $I$  satisfies the positive conjugate condition for all  $C \in \mathcal{C}$ .*

(2) *If  $I(f) = -(C) \int -f d\mu_I^-$  for all  $f \in K$ , then  $I$  satisfies the negative conjugate condition for all  $C \in \mathcal{C}$ .*

## 5 Proof of Theorem 4.4

In this section, the proof of Theorem 4.4 (1) is shown. The proof of Theorem 4.4 (2) is much the same.

Let  $\epsilon > 0$  and  $C \in \mathcal{C}$ .

First suppose that a c.m. functional  $I$  satisfies the positive conjugate condition for every compacc set  $C$ . That is, there exists a positive real number  $M$  such that  $\forall \epsilon > 0$ ,  $\exists f_1, f_2 \in K_1$ ,  $1_C \leq g_1 \leq f_1$  and  $f_2 \leq g_2 \leq 1_{C^c}$  with  $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$  imply

$$M - I(g_2) - \epsilon < -I(-g_1) < M - I(g_2) + \epsilon \quad (1)$$

for  $g_1, g_2 \in K_1$ .

Since  $\mu_I^-$  is regular, there exists an open set  $O$  such that  $C \subset O$  and

$$\mu_I^-(C) + \epsilon \geq \mu_I^-(O). \quad (2)$$

Using Uryson's lemma, there exists  $h_1 \in K_1^+$  such that  $1_C \leq h_1 \leq 1_O$ . Since  $1_C \leq f_1$ , we may suppose that  $f_1 \geq h_1$ . It follows from Lemma 4.3 that

$$\mu_I^-(C) \leq -I(-h_1). \quad (3)$$

Since  $\text{supp}(h_1) \subset O$ , we have

$$\mu_I^-(O) \geq -I(-h_1) \quad (4)$$

from the definition of  $\mu_I^-$ . Then it follows from (2) and (4) that

$$\mu_I^-(C) + \epsilon \geq -I(-h_1). \quad (5)$$

Since  $C^c$  is an open set, it follows from the definition of the induced regular fuzzy measure  $\mu_I^+$  that there exists  $h_2 \in K_1^+$  such that  $\text{supp}(h_2) \subset C^c$  and

$$I(h_2) \geq \mu_I^+(C^c) - \epsilon. \quad (6)$$

We may suppose that  $f_2 \leq h_2 \leq 1_{C^c}$ . Then applying (5) and (6), we have

$$\mu_I^-(C) + \epsilon \geq M - I(h_2) - \epsilon. \quad (7)$$

Since we have  $I(h_2) \leq \mu_I^+(C^c)$  from  $\text{supp}(h_2) \subset C^c$ , we have

$$\mu_I^-(C) + \epsilon \geq M - \mu_I^+(C^c) - \epsilon. \quad (8)$$

Since  $I$  satisfies the conjugate condition for  $\emptyset$ , we have  $M = \mu_I^+(X)$ . Therefore we have

$$2\epsilon \geq (\mu_I^+)^c(C) - \mu_I^-(C) \quad (9)$$

from (8).

On the other hand, it follows from (1),(2) and (6) that

$$\begin{aligned}
 -I(-h_1) &\leq M - I(h_0) + \epsilon \\
 &\leq M - (\mu_I^+(C^c) - \epsilon) + \epsilon \\
 &\leq (\mu_I^+)^c(C) + 2\epsilon.
 \end{aligned}$$

Therefore we have

$$|\mu_I^-(C) - (\mu_I^+)^c(C)| \leq 2\epsilon.$$

Since  $\epsilon$  is an arbitrary, we have  $\mu_I^-(C) = (\mu_I^+)^c(C)$ .

Next suppose that  $\mu_I^-(C) = (\mu_I^+)^c(C)$ . Define  $M = \mu_I^+(X)$ . Then it follows from the definition of the conjugate of  $\mu_I^-$  that

$$\mu_I^-(C) = M - \mu_I^-(C^c). \quad (10)$$

Since  $\mu_I^-$  is regular, there exists an open set  $O$  such that  $O \supset C$  and

$$\mu_I^-(C) + \epsilon \geq \mu_I^-(O). \quad (11)$$

Using Uryson's lemma, there exists  $f_1 \in K_1^+$  such that  $1_C \leq f_1 \leq 1_O$ . Then for every  $g_1 \in K_1^+$  such that  $1_C \leq g_1 \leq f_1$ , we have

$$\mu_I^-(O) \geq -I(-g_1) \geq \mu_I^-(C) \quad (12)$$

from Lemma 4.3. It follows from the definition of the induced regular fuzzy measure  $\mu_I^+$  that there exists  $f_2 \in K_1^+$  such that  $\text{supp}(f_2) \subset C^c$  and

$$\mu_I^+(C^c) - \epsilon \leq I(f_2). \quad (13)$$

Therefore for every  $g_2 \in K_1^+$  such that  $f_2 \leq g_2 \leq C^c$  and  $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$ , we have

$$\mu_I^+(C^c) - \epsilon \leq I(f_2) \leq I(g_2) \leq \mu_I^+(C^c). \quad (14)$$

It follows from (10),(11) and (12) that

$$M - \mu_I^+(C^c) + \epsilon \geq -(-g_2).$$

Then we have

$$\epsilon \geq -M - I(-g_1) + I(g_2) \quad (15)$$

from (14). On the other hand, it follows from (10) and (14) that

$$I(g_2) + \epsilon \geq M - \mu_I^-(C). \quad (16)$$

Then we have

$$\epsilon \geq M - I(g_2) + I(-g_1) \quad (17)$$

from (12). Therefore we have

$$|I(-g_1) - I(g_2) + M| < \epsilon$$

from (15) and (17). □

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